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RESEARCH ARTICLE

Wormhole in the modified theory of gravity in Kaluza–Klein cosmology

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Abstract
Noether symmetry in the modified theory of gravity with $F(\mathcal{R})$-term in the action in the background of 5-dimensional Kaluza–Klein cosmology yields $F(\mathcal{R}) \propto \mathcal{R}^n$ for arbitrary n. Wormhole is allowed quantum mechanically with both $n = \frac{3}{2}$ and 2. Further wormhole is allowed from the Euclidean field equations for $n = \frac{3}{2}$ only, however for $n = 2$ its existence is not clear. Quantum wormholes are also allowed for arbitrary n with $n > \frac{1}{2}$ except $n = 1$ and $\frac{13}{10}$ from the Wheeler–DeWitt equation in terms of new canonical variables obtained from Noether symmetry, consistent with solution in terms of old variables. We present concept of time in quantum cosmology with use of Noether constant of motion. We also present an idea of transition from wormhole configuration to the inflationary era.

Keywords Modified theory of gravity · Noether symmetry · 5-Dimensional Kaluza–Klein cosmology · Wormhole

1 Introduction

Observation of high redshift of SNIa data [1,2] and CMBR anisotropy of WMAP [3–5] confirm that the present expansion of the universe is accelerated, while at early era it was vacuum dominated. Inflaton field has been invoked to consider early inflationary universe, however fine tuning of potential is required for graceful exit. Further, the

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ΛCDM concordance model, or any other model used to advocate late time acceleration are not completely satisfactory.

The present scenario of late time transition to accelerating expansion is experienced by violating strong energy condition $\rho + 3p \geq 0$. The violation is invoked by a new form of energy, dubbed as dark energy. The violation of energy condition is also necessary to explain early inflation, as well as wormhole configuration. It may be realised in two distinct ways: one may introduce exotic field [6–8] in the Einstein field equation, dubbed as dark energy; or by modifying Einstein's gravity, which leads to the modified theory of gravity or $F(R)$ gravity.

In the phenomenological modification of the Einstein theory of gravity, the quantum corrections were introduced from the fundamentals of physics to improve renormalizability of general relativity. Starobinsky [9,10] first proposed a model of the early inflationary universe based on R^2 term in the action, which is a unique term in the modified theory of gravity. Further, the low energy description of string/M theory gives rise to different curvature terms ($R^{\frac{3}{2}}$, $\frac{1}{R}$, etc.) [11–13]. Substantial amount of work [14–18] has been done in the literature of the modified theory of gravity to get both early inflation and late time acceleration, however none of the model is quite satisfactory to give a viable evolution of the universe.

Wormholes (or instantons) were discussed in many theories [19–25]. Recently wormhole (traversable) in the modified theory of gravity has been discussed in [26,27], however in our discussion we consider with cosmological wormhole in the modified theory of gravity. The wormhole consists of two asymptotic regions connected by a throat governed by the Euclidean field equations. The wormhole solution represents tunneling through the throat from one space to another one without encountering cosmic singularity and the configuration is symmetric about the throat. In continuation of wormhole configuration, Hawking and Page [28] assumed it as a quantum mechanical object governed by the Wheeler–DeWitt (WD) equation satisfying some boundary conditions instead of assuming wormhole simply as a solution of Euclidean field equation. The wormhole boundary conditions [28] is that the wave function is exponentially damped for large three geometries and is regular when three geometry vanishes to zero in the background of Robertson–Walker misuperspace. We further extend the validity of Hawking–Page wormhole boundary condition even in 5-dimensional Kaluza–Klein spacetime. In 5-dimensional Kaluza–Klein spacetime the wave function should be exponentially damped for large 4-spatial geometry and it should be regular in some suitable way when the 4-spatial geometry degenerates. The vanishing of the cosmological constant is not well understood and it requires fine tuning, however in the wormhole scenario one may regularize [24,25,29] the value of the fundamental parameters, as well as one may justify validity of a given theory (quantum cosmology) at the Planck epoch, wherein wormhole is relevant. This motivates us to seek wormhole configuration in the modified theory of gravity.

Recently evolution of the universe in the modified theory of gravity in 5-dimensional Kaluza–Klein spacetime have been dicussed [30–32] with a $F(\mathcal{R})$-term (here, $\mathcal{R}$ represents Ricci scalar for 5-dimensional Kaluza–Klein spacetime throughout this manuscript) in the action, however they did not specify a unique form of $F(\mathcal{R})$. The basic form of $F(\mathcal{R})$ should represent the observed data and in absence of any such form we invoke Noether symmetry [33–46] to find $F(\mathcal{R})$ in 5-dimensional KK spacetime, which yields $F(\mathcal{R}) = F_0\mathcal{R}^n$ for arbitrary n except $n = 1$.

The field equations both in the Lorentzian and Euclidean sections are identical for $\mathcal{R}^n$-action for vanishing three space curvature parameter (i.e. $\kappa = 0$). We further present analytic and numerical solution of wormhole for $n = 3/2$, while in $n = 2$ wormhole solution is not attainable due to complexity, hence we introduce new set of configuration space variables ($Q_i, i = 1, 2, 3$) using Noether symmetry to simplify the equations, which again leads to cumbersome equations, so we skip Euclidean field equations to investigate wormhole configuration.

We further present wormhole solution from the WD-equation (in terms of old configuration space variables) for $\mathcal{R}^n$-action with both $n = 3/2$ and $n = 2$ satisfying Hawking–Page wormhole boundary condition. This shows that the wormhole based on Hawking–Page condition [28] at the Planck epoch is more relevant and appropriate in early universe than the wormhole solution obtained from the Euclidean field equations. We further construct the WD-equation with help of new variables (Q_i) and present the wave function for arbitrary n with $n > \frac{1}{2}$, except $n = 1, \frac{13}{10}$ consistent with wormhole solutions respectively for $n = 3/2$ and $n = 2$.

The time parameter does not appear explicitly in the WD-equation rather the concept of time is inbuilt in it. One can recover time parameter by introducing semi-classical approximation to the WD-equation by establishing Tomonaga–Schwinger type equation, which is a functional Schrödinger equation [47]. In above quantization we recover time parameter equivalent to Q_3 variable by introducing Noether constant [48] of motion. Then resulting quantum equation yields the idea of probability from the continuity equation.

In Sect. 2, we construct Lagrangian for an action with $F(\mathcal{R})$-term introducing Lagrange multiplier in 5-dimensional Kaluza–Klein (KK) spacetime having topology $R^1 \otimes E^3 \otimes S^1$. Further in Sect. 3, we consider Noether symmetry approach [33] to get a suitable form of $F(\mathcal{R})$ in the cosmological point of view which can yield a viable cosmic evolution. In Sect. 4, we write down the field equations and present the solution of the Euclidean field equations for $n = \frac{3}{2}$ and 2. Further in Sect. 4.3, we present the Lagrangian introducing new configuration space variables by virtue of Noether symmetry. In Sect. 5, we present the wave function of the universe from the solution of the WD-equation, further in Sect. 5.3 we present the WD-equation and wormhole solution in terms of new variables. Further in Sect. 5.3.1 a new time parameter is identified and a probabilistic interpretation is given in quantum cosmology. In Sect. 6, we present an idea of transition from a wormhole configuration to a power law inflation. Finally in Sect. 7, we present a brief discussion.

2 Action and the Lagrangian

We consider the action for arbitrary function of $\mathcal{R}$ in the form

$$A = \int d^5x\sqrt{-{}^5g}\; F(\mathcal{R}). \tag{1}$$

The field equations can be obtained by variation of action, however we consider the Lagrange multiplier technique in spatially homogeneous space. We assume 5-dimensional Kaluza–Klein spacetime as

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] + b^2(t)d\xi^2.$$

Now using Lagrange Multiplier technique in view of Kaluza–Klein spacetime of topology $R^1 \otimes E^3 \otimes S^1$, the action (1) can be written as,

$$A = \int d^5x\sqrt{-^5g}\left[F(\mathcal{R}) - \lambda\left\{\mathcal{R} - 2\left(3\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + 3\frac{\dot{a}\dot{b}}{ab} + 3\frac{\dot{a}^2}{a^2}\right)\right\}\right], \tag{2}$$

where $\mathcal{R} = 2\left(3\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + 3\frac{\dot{a}\dot{b}}{ab} + 3\frac{\dot{a}^2}{a^2}\right)$, $a(t)$ is the scale factor of external space, $b(t)$ is the scale factor of internal space and $\mathcal{R}$ is the Ricci curvature of the 5-dimensional space time. Varying the action with respect to $\mathcal{R}$ we get $\lambda = F_{,\mathcal{R}}$. Hence the action takes the form

$$A = \int L dt + \Sigma_s, \tag{3}$$

where, the Lagrangian is

$$\begin{aligned} L = a^3b\Big(F(\mathcal{R}) - \mathcal{R}F_{,\mathcal{R}}\Big) - 6a\dot{a}^2bF_{,\mathcal{R}} - 6a^2\dot{a}\dot{b}F_{,\mathcal{R}} \\ - 6a^2b\dot{a}F_{,\mathcal{R}\mathcal{R}}\dot{\mathcal{R}} - 2a^3\dot{b}F_{,\mathcal{R}\mathcal{R}}\dot{\mathcal{R}}, \end{aligned} \tag{4}$$

and $\Sigma_s = 2\left[3a^2b\dot{a}F_{,\mathcal{R}} + a^3\dot{b}F_{,\mathcal{R}}\right]$ is the surface term, and one can get rid of Σ_s with proper choice of action A. A comma (,) stands for partial derivative with respect to the variable at the subscript. The field equations can be obtained from the Lagrangian for the configuration space variables a, b and $\mathcal{R}$. In the modified theory of gravity the functional form $F(\mathcal{R})$ are prescribed to justify the observational results. In absence of prior knowledge of it we follow the Noether symmetry approach to cosmology [33,34]. Thus to determine $F(\mathcal{R})$ we follow earlier work [49].

3 Noether symmetry

Noether symmetry is allowed when the action is invariant under certain transformation in the configuration space. Mathematically Noether symmetry is allowed with existence of $\pounds_{\mathbf{X}}L = 0$, where $\pounds_{\mathbf{X}}L$ is the Lie derivative of the Lagrangian L with respect to the vector field $\mathbf{X}$, where

$$\begin{aligned} \mathbf{X} = \alpha(a, b, \mathcal{R})\frac{\partial}{\partial a} + \beta(a, b, \mathcal{R})\frac{\partial}{\partial b} + \gamma(a, b, \mathcal{R})\frac{\partial}{\partial \mathcal{R}} + \dot{\alpha}(a, b, \mathcal{R})\frac{\partial}{\partial \dot{a}} \\ + \dot{\beta}(a, b, \mathcal{R})\frac{\partial}{\partial \dot{b}} + \dot{\gamma}(a, b, \mathcal{R})\frac{\partial}{\partial \dot{\mathcal{R}}}. \end{aligned} \tag{5}$$

To study the existence of Noether symmetry for arbitrary $F(\mathcal{R})$ we assume that the coefficients of $\dot{a}^2,\ \dot{b}^2,\ \dot{\mathcal{R}}^2,\ \dot{a}\dot{b},\ \dot{a}\dot{\mathcal{R}},\ \dot{b}\dot{\mathcal{R}}$ along with the remaining term to be zero in the expression $\pounds_{\mathbf{X}} L = 0$ as usual [49], so we have

$$\left(\gamma ab + \gamma_{,a}a^2 b\right) F_{,\mathcal{R}\mathcal{R}} + \left(\alpha b + \beta a + 2\alpha_{,a}abF_{,\mathcal{R}} + \beta_{,a}a^2\right) F_{,\mathcal{R}} = 0, \tag{6}$$

$$\gamma_{,b}aF_{,\mathcal{R}\mathcal{R}} + 3\alpha_{,b}F_{,\mathcal{R}} = 0, \tag{7}$$

$$(3\alpha_{,\mathcal{R}}b + \beta_{,\mathcal{R}}a)F_{,\mathcal{R}\mathcal{R}} = 0, \tag{8}$$

$$\left(3\gamma a + \gamma_{,a}a^2 b + 3\gamma_{,b}ab\right) F_{,\mathcal{R}\mathcal{R}} + \left(6\alpha + 3\alpha_{,a}a + 6\alpha_{,b}b + 3\beta_{,b}a\right) F_{,\mathcal{R}} = 0, \tag{9}$$

$$\begin{aligned} &3\gamma abF_{,\mathcal{R}\mathcal{R}\mathcal{R}} + \left(6\alpha b + 3\beta a + 3\alpha_{,a}ab + \beta_{,a}a^2 + 3\gamma_{,\mathcal{R}}ab\right) F_{,\mathcal{R}\mathcal{R}} \\ &\quad + \left(6\alpha_{,\mathcal{R}}b + 3\beta_{,\mathcal{R}}a\right) F_{,\mathcal{R}} = 0, \end{aligned} \tag{10}$$

$$\gamma aF_{,\mathcal{R}\mathcal{R}\mathcal{R}} + \left(3\alpha + 3\alpha_{,b}b + \beta_{,b}a + \gamma_{,\mathcal{R}}a\right) F_{,\mathcal{R}\mathcal{R}} + 3\alpha_{,\mathcal{R}}F_{,\mathcal{R}} = 0, \tag{11}$$

$$\gamma ab\mathcal{R}F_{,\mathcal{R}\mathcal{R}} - (3\alpha b + \beta a)(F - \mathcal{R}F_{,\mathcal{R}}) = 0. \tag{12}$$

The available conditions for existence of Noether symmetry by solving (6)–(12) are

$$F(\mathcal{R}) = F_0\mathcal{R}^n, \quad \alpha = \frac{B_{10}a}{3nF_0}, \quad \beta = -\frac{B_{10}b}{nF_0}, \quad \gamma = 0, \tag{13}$$

where, n is arbitrary constant and F_0 and B_{10} are constants of integration. The Noether constant of motion is

$$\Sigma = \sum_i \alpha_i \frac{\partial L}{\partial \dot{q}_i} = \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) a^3 b\mathcal{R}^{n-1}. \tag{14}$$

It is interesting to note that the Noether symmetry in the modified theory of gravity in 4-dimension allows $F(R) \propto R^{3/2}$ [46] only, whereas in 5-dimensional Kaluza–Klein spacetime there are no restrictions on the exponent n from the Noether symmetry.

4 Field equations for $F(\mathcal{R}) = F_0\mathcal{R}^n$ and solution for $n = \frac{3}{2}$ and $n = 2$

Now to study evolution of the universe the field equations corresponding to $F(\mathcal{R}) = F_0\mathcal{R}^n$ are

$$\begin{aligned} &2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}\dot{b}}{ab} + \frac{\ddot{b}}{b} + \left(2\frac{\dot{a}\dot{\mathcal{R}}}{a\mathcal{R}} + \frac{\ddot{\mathcal{R}}}{\mathcal{R}} + \frac{\dot{b}\dot{\mathcal{R}}}{b\mathcal{R}}\right)(n-1) \\ &\quad + (n-1)(n-2)\frac{\dot{\mathcal{R}}^2}{\mathcal{R}^2} + \frac{(1-n)}{2n}\mathcal{R} = 0, \end{aligned} \tag{15}$$

$$3\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)+\left(3\frac{\dot{a}\dot{\mathcal{R}}}{a\mathcal{R}}+\frac{\ddot{\mathcal{R}}}{\mathcal{R}}\right)(n-1)$$
$$+(n-1)(n-2)\frac{\dot{\mathcal{R}}^2}{\mathcal{R}^2}+\frac{(1-n)}{2n}\mathcal{R}=0 \tag{16}$$

and corresponding Hamiltonian constraint equation (for lapse function $N=1$) is

$$3\left(\frac{\dot{a}^2}{a^2}+\frac{\dot{a}\dot{b}}{ab}\right)+\left(3\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)\frac{\dot{\mathcal{R}}}{\mathcal{R}}(n-1)+\frac{(1-n)}{2n}\mathcal{R}=0. \tag{17}$$

Further using (15) and (16) we obtain the same constant of motion Σ, which shows intrinsic presence of Noether symmetry in action with $F(\mathcal{R})=F_0\mathcal{R}^n$ for arbitrary n in Kaluza–Klein cosmology. Now we consider the solution of the Euclidean field equation to study wormhole or instanton configuration.

Instanton solution
An instanton is a classical solution to the equations of motion with a finite, non-zero action either in classical field theory or in quantum theory on a Euclidean spacetime. The solution of the field equations is difficult in general, so we consider the action with $F(\mathcal{R})=F_0\mathcal{R}^n$ for $n=\frac{3}{2}$ and 2 assuming some restriction on the internal space to get wormhole or instanton solution.

4.1 Solution for $F(\mathcal{R})=\mathcal{R}^{\frac{3}{2}}$

The field equation in the Euclidean background considering Wick rotation $t\to -i\tau$ can be obtained from (15)–(17). Now the field Eq. (17) with the definition of Ricci scalar $\mathcal{R}$ gives

$$\frac{18\left(a''\right)^2}{a^2}+\frac{18\left(a'\right)^4}{a^4}-\left(\frac{9a^{(3)}b'}{ab}-\frac{12a''b''}{ab}+\frac{9a''\left(b'\right)^2}{ab^2}+\frac{9b^{(3)}a'}{ab}\right.$$
$$-\frac{9a'\left(b'\right)^3}{ab^3}+\frac{6a'b'b''}{ab^2}+\frac{27\left(a'\right)^3b'}{a^3b}$$
$$\left.+\frac{33\left(a'\right)^2b''}{a^2b}+\frac{54a'a''b'}{a^2b}-\frac{2\left(b''\right)^2}{b^2}-\frac{3\left(b'\right)^2b''}{b^3}+\frac{3b^{(3)}b'}{b^2}\right)$$
$$-\frac{45\left(a'\right)^2a''}{a^3}-\frac{27a^{(3)}a'}{a^2}=0, \tag{18}$$

Further a variation equation is

$$\frac{216b''\left(a'\right)^4}{a^4b}-\frac{81\left(b'\right)^2\left(a'\right)^4}{a^4b^2}+\frac{54\left(b'\right)^3\left(a'\right)^3}{a^3b^3}+\frac{162b'a''\left(a'\right)^3}{a^4b}$$
$$+\frac{54b'b''\left(a'\right)^3}{a^3b^2}+\frac{216a^{(3)}\left(a'\right)^3}{a^4}$$

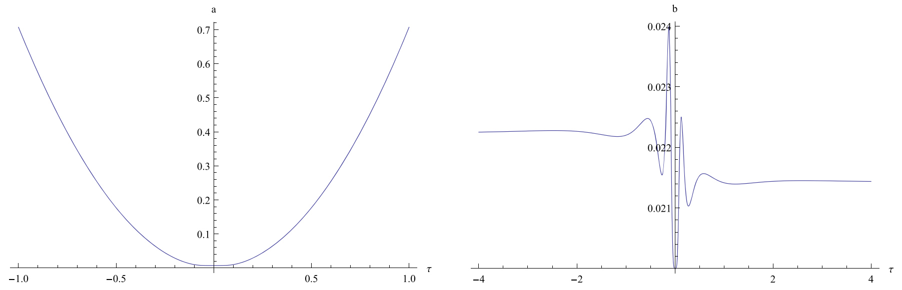
$$
\begin{aligned}
&+\frac{126b^{(3)}\left(a'\right)^3}{a^3b}+\frac{27\left(b'\right)^4\left(a'\right)^2}{a^2b^4}+\frac{3\left(b''\right)^2\left(a'\right)^2}{a^2b^2}+\frac{324\left(b'\right)^2a''\left(a'\right)^2}{a^3b^2}\\
&+\frac{36\left(b'\right)^2b''\left(a'\right)^2}{a^2b^3}+\frac{162a''b''\left(a'\right)^2}{a^3b}\\
&+\frac{270b'a^{(3)}\left(a'\right)^2}{a^3b}+\frac{90b'b^{(3)}\left(a'\right)^2}{a^2b^2}+\frac{54a^{(4)}\left(a'\right)^2}{a^3}+\frac{18b^{(4)}\left(a'\right)^2}{a^2b}\\
&-\frac{81\left(a''\right)^2\left(a'\right)^2}{a^4}+\frac{324b'\left(a''\right)^2a'}{a^3b}\\
&+\frac{12b'\left(b''\right)^2a'}{ab^3}+\frac{54\left(b'\right)^3a''a'}{a^2b^3}+\frac{18\left(b'\right)^3b''a'}{ab^4}+\frac{216b'a''b''a'}{a^2b^2}\\
&+\frac{162\left(b'\right)^2a^{(3)}a'}{a^2b^2}+\frac{54a''a^{(3)}a'}{a^3}\\
&+\frac{72a''b^{(3)}a'}{a^2b}+\frac{12b''b^{(3)}a'}{ab^2}+\frac{54b'a^{(4)}a'}{a^2b}+\frac{18b'b^{(4)}a'}{ab^2}-\frac{18b''a^{(3)}a'}{a^2b}\\
&+\frac{162\left(a''\right)^3}{a^3}+\frac{2\left(b''\right)^3}{b^3}+\frac{3\left(b'\right)^2\left(b''\right)^2}{b^4}\\
&+\frac{78a''\left(b''\right)^2}{ab^2}+\frac{270\left(a''\right)^2b''}{a^2b}+\frac{18\left(b'\right)^2a''b''}{ab^3}+\frac{54b'a''a^{(3)}}{a^2b}+\frac{54b'b''a^{(3)}}{ab^2}\\
&+\frac{54a''a^{(4)}}{a^2}+\frac{18b''a^{(4)}}{ab}\\
&+\frac{18a''b^{(4)}}{ab}+\frac{6b''b^{(4)}}{b^2}-\frac{27\left(a^{(3)}\right)^2}{a^2}-\frac{18a^{(3)}b^{(3)}}{ab}-\frac{3\left(b^{(3)}\right)^2}{b^2}\\
&-\frac{36b'a''b^{(3)}}{ab^2}-\frac{81\left(b'\right)^2\left(a''\right)^2}{a^2b^2}=0
\end{aligned}
\tag{19}
$$

and b variation equation may be obtained in the same way, where an over head prime denotes derivative with respect to τ. It is interesting to note that the field equations are same both in the Lorentzian and Euclidean signature. Analytic solution of above equations is not possible in general, however for static internal space ($b = b_0$) integration of (18) gives

$$a^2(\tau) = a_0^2 + \left(A^{\frac{1}{4}}\tau + B\right)^4, \tag{20}$$

where A, B, a_0 are integration constants, and one can determine the constants with knowledge of initial conditions. Interestingly all the field equations are satisfied by the solution. In spatially homogeneous and isotropic space by use of time translation symmetry we can put $B = 0$, which yields

$$a^2(\tau) = a_0^2 + A\tau^4 \tag{21}$$



(a) $a(\tau)$ versus τ **(b)** $b(\tau)$ versus τ

Fig. 1 The evolution of $a(\tau)$ in (**a**) and in $b(\tau)$ (**b**) with τ are given by numerical solution of the field Eqs. (18) and (19) using expression of $\mathcal{R}$ with the initial conditions $a[0.01] = 0.00707, a'[0.01] = 0.00014, a''[0.01] = 0.04242, a'''[0.01] = 8.4823$ and $b[.01] = .02, b'[.01] = -.0001, b''[.01] = 2.10^{-7}, b'''[.01] = -244.77$

A reflection symmetry $a(\tau) = a(-\tau)$ is obtained about $\tau = 0$ for this solution with $B = 0$. Thus the solution (21) represents a wormhole configuration and the radius at the throat is a_0 (at $\tau = 0$) for a static internal space.

Further a numerical solution of (18) and (19) with suitable initial conditions (given in Fig. 1) yield wormhole configuration in the external space, while the internal scale factor at the throat of the wormhole is minimum and fluctuating, thereafter $b(\tau)$ decreases and attains a fixed value without dimensional reduction. The radius at the throat of the wormhole depends on the initial conditions.

4.2 Solution for $F(\mathcal{R}) = \mathcal{R}^2$

The field Eq. (17) in the Euclidean background is

$$\frac{3a^{(3)}b'}{ab} - \frac{3a''b''}{ab} + \frac{3a''\left(b'\right)^2}{ab^2} - \frac{9\left(a''\right)^2}{2a^2} + \frac{3b^{(3)}a'}{ab} - \frac{3a'\left(b'\right)^3}{ab^3} - \frac{27\left(a'\right)^4}{2a^4} - \frac{6\left(a'\right)^3 b'}{a^3 b} + \frac{9\left(a'\right)^2 b''}{a^2 b} + \frac{12a'a''b'}{a^2 b} - \frac{15\left(a'\right)^2\left(b'\right)^2}{2a^2b^2} + \frac{9\left(a'\right)^2 a''}{a^3} + \frac{9a^{(3)}a'}{a^2} - \frac{\left(b''\right)^2}{2b^2} - \frac{\left(b'\right)^2 b''}{b^3} + \frac{b^{(3)}b'}{b^2} = 0, \tag{22}$$

and the a variation equation is

$$\frac{6a^{(4)}}{a} + \frac{12a^{(3)}b'}{ab} + \frac{16a''b''}{ab} - \frac{6a''\left(b'\right)^2}{ab^2} + \frac{9\left(a''\right)^2}{a^2} + \frac{10b^{(3)}a'}{ab} + \frac{6a'\left(b'\right)^3}{ab^3} - \frac{12\left(a'b'b''\right)}{ab^2} + \frac{9\left(a'\right)^4}{a^4} - \frac{12\left(a'\right)^3 b'}{a^3 b}$$

$$+\frac{2\left(a'\right)^2 b''}{a^2 b}+\frac{6a'a''b'}{a^2 b}-\frac{3\left(a'\right)^2\left(b'\right)^2}{a^2b^2}-\frac{36\left(a'\right)^2 a''}{a^3}+\frac{12a^{(3)}a'}{a^2}+\frac{2b^{(4)}}{b}$$
$$-\frac{\left(b''\right)^2}{b^2}+\frac{2\left(b'\right)^2 b''}{b^3}-\frac{2b^{(3)}b'}{b^2}=0 \tag{23}$$

and b variation equation may be obtained in the same way. In this case the field equations are same both in the Lorentzian and Euclidean signature like earlier $n = 3/2$ case. The solution of the field equations in general is not possible. However, for a static background of the internal space we have

$$-\frac{a\,a''}{a'^2}=\frac{a_0^3-a^3}{a_0^3+a^3}, \tag{24}$$

where a_0^3 is a constant. An integration of it gives

$$\frac{a^2}{a_0^2}\,\text{Hypergeometric2F1}\left[\frac{2}{3},\frac{2}{3},\frac{5}{3},-\frac{a^3}{a_0^3}\right]=N\tau, \tag{25}$$

where N is a constant. This does not allow a wormhole configuration in static internal background. Further plausible wormhole configuration from numerical solution is not attainable even in non-static internal background. Analytical solution of the field equations is really cumbersome, so we consider an alternative way to alleviate the problem. The Noether symmetry naturally yields cyclic coordinates and one may expect solution of the field equations without much hindrance in terms of the new variables.

4.3 New configuration space variables from Noether symmetry and the field equations

Let us introduce $\{Q_1, Q_2, Q_3\}$ as the new set of configuration space variables replacing the old variables $\{a, b, \mathcal{R}\}$. The cyclic coordinate in the Noether symmetry can be obtained from the generator of the transformation (i.e. $\mathbf{X}$). Let us assume Q_3 is the cyclic coordinate, then $\mathbf{X}=\frac{\partial}{\partial Q_3}$ which yields

$$\Sigma_i\alpha_i\frac{\partial Q_k}{\partial q_i}=0\quad\text{and}\quad\Sigma_i\alpha_i\frac{\partial Q_3}{\partial q_i}=1, \tag{26}$$

where $\alpha_i\equiv(\alpha,\beta,\gamma)$, $Q_k\equiv(Q_1,Q_2)$. Hence the set of Eqs. (26) for (13) read as

$$\frac{a}{3}\frac{\partial Q_1}{\partial a}-b\frac{\partial Q_1}{\partial b}=0, \tag{27}$$
$$\frac{a}{3}\frac{\partial Q_2}{\partial a}-b\frac{\partial Q_2}{\partial b}=0, \tag{28}$$
$$\frac{a}{3}\frac{\partial Q_3}{\partial a}-b\frac{\partial Q_3}{\partial b}=\frac{nF_0}{B_{10}}. \tag{29}$$

Solution of the above set of equations is given by

$$Q_1 = a^{3\chi} b^{\chi} \mathcal{R}^{\frac{1}{\sigma}}, \quad Q_2 = a^{\frac{1}{\rho}} b^{\frac{1}{3\rho}} \quad \text{and} \quad Q_3 = Log\Big[a^3 b^{1-\frac{1}{\delta}}\Big], \tag{30}$$

where ρ, χ, σ and $\delta = \frac{B_{10}}{nF_0}$ are constant. Now in view of new coordinates Lagrangian takes the form

$$\begin{aligned} L = 6n Q_1^{\sigma(n-1)} Q_2^{3\rho-3\rho\sigma\chi(n-1)} \Bigg[& \frac{1-n}{6n} Q_1^{\sigma} Q_2^{-3\chi\rho\sigma} \\ & + \rho^2 \Big\{ 3(n-1)\sigma\chi - (1+\delta-2\delta^2) \Big\} \frac{\dot{Q}_2^2}{Q_2^2} \\ & + \frac{2}{9}\delta^2 \dot{Q}_3^2 - (n-1)\rho\sigma \frac{\dot{Q}_2 \dot{Q}_1}{Q_2 Q_1} + \frac{\rho\delta}{3}(1-4\delta) \frac{\dot{Q}_2 \dot{Q}_3}{Q_2} \Bigg]. \end{aligned} \tag{31}$$

Now choosing $4\delta = 1$ we can simplify the field equations, however analytical solution is quite non-trivial. Investigation of the field equations in the new configuration space $\{Q_1, Q_2, Q_3\}$ variables using Noether symmetry does not yield any solution. So we consider quantization of the Hamilton constraint equation to study possible wormhole configuration in terms of the wave function of the universe representing Hawking–Page [28] wormhole boundary condition.

5 Wheeler–DeWitt equation in the Kaluza–Klein cosmology in terms of mini-superspace variables a, b, $\mathcal{R}$

The canonical quantisation can be performed in the Kaluza–Klein mini-superspace and hence the solution of the wave function can be obtained. Now we consider the Wheeler–DeWitt equation and its solution both for $\mathcal{R}^{\frac{3}{2}}$ and $\mathcal{R}^2$ terms separately in the the action.

5.1 Wheeler–DeWitt equation and its solution for $\mathcal{R}^{\frac{3}{2}}$-term

The Hamiltonian constraint equation from (4) for $\mathcal{R}^{\frac{3}{2}}$-action yields

$$\begin{aligned} a^2 p_a^2 + 9b^2 p_b^2 + 36\mathcal{R}^2 p_{\mathcal{R}}^2 - 6ab p_a p_b - 12a\mathcal{R} p_a p_{\mathcal{R}} \\ -12b\mathcal{R} p_b p_{\mathcal{R}} - 36F_0^2 a^6 b^2 \mathcal{R}^2 = 0. \end{aligned} \tag{32}$$

The Wheeler–DeWitt equation for above equation can be formulated as

$$\begin{aligned} a^2 \left[\frac{\partial^2 \psi}{\partial a^2} + \frac{p_1}{a} \frac{\partial \psi}{\partial a} \right] + 9b^2 \left[\frac{\partial^2 \psi}{\partial b^2} + \frac{p_2}{b} \frac{\partial \psi}{\partial b} \right] + 36\mathcal{R}^2 \left[\frac{\partial^2 \psi}{\partial \mathcal{R}^2} + \frac{p_3}{\mathcal{R}} \frac{\partial \psi}{\partial \mathcal{R}} \right] \\ - 6ab \frac{\partial^2 \psi}{\partial a \partial b} - 12a\mathcal{R} \frac{\partial^2 \psi}{\partial a \partial \mathcal{R}} - 12b\mathcal{R} \frac{\partial^2 \psi}{\partial b \partial \mathcal{R}} + \frac{36}{\hbar^2} F_0^2 a^6 b^2 \mathcal{R}^2 \psi = 0, \end{aligned} \tag{33}$$

where p_1, p_2, p_3 are the operator ordering parameters. A simple solution of the above equation is

$$\psi(a, b, \mathcal{R}) = e^{ra^3 b\mathcal{R}}, \tag{34}$$

where $r = \pm\frac{F_0}{\hbar}\sqrt{3}$ and $p_3 = \frac{5}{3} - \frac{2p_1}{12} - \frac{p_2}{4}$. It is important to note that the integration constant F_0 in (13) is arbitrary and the evolution of the classical field equations are independent of the signature of F_0. However, the quantum dynamics, or the wave function ψ depends on F_0. Introducing r as negative in (34) the wave function is exponentially damped for large 4-(spatial) geometries, while at vanishing four geometry it is regular. Thus the wave function (34) satisfies the Hawking–Page wormhole boundary condition even in 5-dimensional Kaluza–Klein cosmology, where the spatial geometry is 4-dimensional and the volume of proper 4-(spatial) geometry is proportional to $a^3 b$. Thus the action in the modified theory of gravity with $\mathcal{R}^{\frac{3}{2}}$ allows quantum mechanical wormhole configuration in the Kaluza–Klein cosmology.

5.2 Wheeler–DeWitt equation and its solution for $\mathcal{R}^2$-term

The Hamiltonian constraint equation from (4) for $\mathcal{R}^2$-action also gives

$$a^2 p_a^2 + 9b^2 p_b^2 + 9\mathcal{R}^2 p_{\mathcal{R}}^2 - 6ab p_a p_b - 6a\mathcal{R} p_a p_{\mathcal{R}} \\ - 6b\mathcal{R} p_b p_{\mathcal{R}} - 96F_0^2 a^6 b^2 \mathcal{R}^3 = 0. \tag{35}$$

Now the Wheeler–DeWitt equation from above equation can be formulated as

$$a^2\left[\frac{\partial^2\psi}{\partial a^2} + \frac{p_1}{a}\frac{\partial\psi}{\partial a}\right] + 9b^2\left[\frac{\partial^2\psi}{\partial b^2} + \frac{p_2}{b}\frac{\partial\psi}{\partial b}\right] \\ +9\mathcal{R}^2\left[\frac{\partial^2\psi}{\partial\mathcal{R}^2} + \frac{p_3}{\mathcal{R}}\frac{\partial\psi}{\partial\mathcal{R}}\right] - 6ab\frac{\partial^2\psi}{\partial a\partial b} - 6a\mathcal{R}\frac{\partial^2\psi}{\partial a\partial\mathcal{R}} \\ -6b\mathcal{R}\frac{\partial^2\psi}{\partial b\partial\mathcal{R}} + \frac{96}{\hbar^2}F_0^2 a^6 b^2 \mathcal{R}^3\psi = 0, \tag{36}$$

where p_1, p_2, p_3 are the operator ordering parameters. A simple solution of (36) analogous to (33) is

$$\psi(a, b, \mathcal{R}) = e^{r_0 a^3 b\mathcal{R}^{\frac{3}{2}}}, \tag{37}$$

where $r_0 = \pm\frac{8F_0}{\hbar}\sqrt{\frac{2}{21}}$ and $p_3 = \frac{55}{18} - \frac{2p_1}{9} - \frac{2p_2}{3}$. Now for negative value of r_0, the wave function is exponentially damped for large 4-geometry, while at vanishing spatial geometry the wave function is finite, hence it satisfies the Hawking–Page wormhole boundary condition following our earlier argument. Thus the modified theory of gravity with $\mathcal{R}^2$ in the Kaluza–Klein cosmology allows the configuration of the wormhole from the Wheeler–DeWitt equation.

5.3 Wheeler–DeWitt equation in view of new variables and its solution for R^n-term

The Hamilton constraint equation from (31) is

$$P_{Q_3}^2 - uQ_1^2 P_{Q_1}^2 - vQ_1 Q_2 P_{Q_1} P_{Q_2} + \frac{n}{3}V = 0, \tag{38}$$

where $u = \frac{(n-1)r-\frac{3}{8}}{6\sigma^2(n-1)^2}$, $v = \frac{1}{18\rho\sigma(n-1)}$, $V = (1-n)Q_1^{\sigma(2n-1)}Q_2^{6\rho-3\rho r(2n-1)}$ and P_{Q_1}, P_{Q_2}, P_{Q_3} are the canonical momenta corresponding to the coordinates Q_1, Q_2, Q_3. In study of Noether symmetry $\frac{\partial}{\partial Q_3}$ is the generator of the transformation in Noether symmetry and P_{Q_3} is the corresponding Noether constant (i.e. Σ) of motion in the classical system. The canonical quantization of Eq. (38) in general leads the Wheeler–DeWitt equation in terms of mini-superspace variables as

$$\begin{aligned} &3\frac{\partial^2\psi}{\partial Q_3^2} - 3uQ_1^2\left(\frac{\partial^2\psi}{\partial Q_1^2} + \frac{p}{Q_1}\frac{\partial\psi}{\partial Q_1}\right) \\ &- 3vQ_1Q_2\frac{\partial^2\psi}{\partial Q_1\partial Q_2} - \frac{n}{\hbar^2}(1-n)Q_1^{\sigma(2n-1)}Q_2^{6\rho-3\rho r(2n-1)}\psi = 0, \end{aligned} \tag{39}$$

where p is the operator ordering parameter and $r = \sigma\chi$. Now the wave function from (39) gives

$$\psi(a, b, \mathcal{R}) = \psi_0 e^{-A\, a^3 b\, \mathcal{R}^{\frac{2n-1}{2}}} \tag{40}$$

transforming (Q_1, Q_2, Q_3) in terms of $(a, b, \mathcal{R})$, where $A = \frac{8}{\hbar}\sqrt{\frac{n(n-1)^3}{(2n-1)(10n-13)}}$ and the parameter p is determined by

$$\sigma(10n-13) + 2(p-1)\left[(n-1)r - \frac{3}{8}\right] = 0.$$

It is noted that the solution (40) is valid for wide range of values of n with $n > \frac{1}{2}$, except $n = 1$ and $\frac{13}{10}$. Interestingly (40) leads to (34) and (37) respectively for $n = 3/2$ and $n = 2$. The wave function (40) satisfies the Hawking–Page wormhole boundary condition, so wormholes are allowed quantum mechanically both for $\mathcal{R}^{3/2}$ and $\mathcal{R}^2$.

5.3.1 Idea of time in quantum cosmology from Noether symmetry

The WD equation is explicit independent of time, rather it is inbuilt through the configuration space variable Q_1, Q_2, Q_3. The concept of time is a classical one, and we can recover it through the semi-classical approximation [50] to the WD equation. Instead of semi-classical approximation we can recover it by use of Noether symmetry introducing

$$P_{Q_3}^2 \to \widehat{P_{Q_3}^2} = \Sigma\,\widehat{P_{Q_3}} \tag{41}$$

in (38). Then canonical quantization of (38) yields

$$i\hbar\frac{\partial\psi}{\partial Q_3} = -\hbar^2\frac{u}{\Sigma}\left(\frac{\partial^2\psi}{\partial X^2} + \frac{p}{X}\frac{\partial\psi}{\partial X}\right) - \hbar^2\frac{v}{\Sigma}\frac{\partial^2\psi}{\partial X\partial Y} + \frac{n}{3\Sigma}V(X,Y)\psi, \tag{42}$$

where $X = \sigma\ln Q_1$, $Y = \rho\ln Q_2$, $V(X,Y) = (n-1)e^{(2n-1)X+3(2-2rn+r)}$ and the wave function $\psi = \psi(Q_3, X, Y)$. The Eq. (41) takes a look of Schrödinger equation assuming Q_3 as the time. Now taking complex conjugate of (42) and multiplying it by ψ from left and with a simplification for $p = 0$ we have

$$\frac{\partial\rho}{\partial Q_3} = -\vec{\nabla}\cdot\vec{J}, \tag{43}$$

where $\rho = \psi^*\psi$ is the probability density, an asterisk denotes complex conjugation, $\vec{J} \equiv \left(j_\eta, j_\xi, j_X\right)$ is the current density and $\vec{\nabla} \equiv \left(\frac{\partial}{\partial\eta}, \frac{\partial}{\partial\xi}, \frac{\partial}{\partial X}\right)$, $X = \eta + i\xi$, $Y = \eta - i\xi$, $\frac{\partial}{\partial\eta} = \frac{\partial}{\partial X} + \frac{\partial}{\partial Y}$, $\frac{\partial}{\partial\xi} = \frac{\partial}{\partial X} + \frac{\partial}{\partial Y}$ and

$$j_\eta = \frac{v\hbar}{4i\Sigma}\left[\psi^*\frac{\partial\psi}{\partial\eta} - \psi\frac{\partial\psi^*}{\partial\eta}\right],\quad j_\xi = \frac{v\hbar}{4i\Sigma}\left[\psi^*\frac{\partial\psi}{\partial\xi} - \psi\frac{\partial\psi^*}{\partial\xi}\right],$$
$$j_X = \frac{u\hbar}{i\Sigma}\left[\psi^*\frac{\partial\psi}{\partial X} - \psi\frac{\partial\psi^*}{\partial X}\right]. \tag{44}$$

The Eq. (43) seems to be a continuity equation assuming increasing Q_3 to the direction of evolution of the system. It is to note that the operator $\frac{\partial}{\partial Q_3}$ is the generator of the Noether symmetry and it is a tangent to the Q_3 line. So idea of time appears in quantum cosmology from Noether symmetry. The WD equation in general does not allow an equation like (43), however in some modified theory of gravity [51,52] continuity equation may be obtain.

6 Possible transition from wormhole configuration to power law inflationary era

In the evolutionary scenario the Planck epoch was occurred at $t \le 10^{-43}$ s, wherein the dynamics is governed by quantum gravity in Euclidean space. Further the inflationary era is determined by the quantum field theory in curved spacetime, which continued till $t \le 10^{-32}$ s; and thereafter usual era of radiation, matter dominated, etc proceeds. In the semi-classical or classical domain, Lorentzian spacetime is relevant. Since the field equations are same, both in the Lorentzian and Euclidean signature for vanishing three space curvature, so the solution (21) for $n = \frac{3}{2}$ turns to the identical form under Wick rotation $\tau \to it$, i.e.

$$a^2(t) = At^4 + a_0^2 \quad \text{for} \quad b = b_0, \tag{45}$$

where the scale factor is expressed in terms of the Lorentzian time t. The Eq. (45) gives rise power law inflation (i.e. $a \sim t^m,\ m > 1$) at large t, when the contribution of a_0^2 is negligible with respect to the first term. The solution (45) is similar to those presented by Capozzeillo et al. [53] and Sarkar et al. [54] in the Robertson–Walker background in 4-dimension. The idea of wormhole according to (21) appears in the very beginning near the Planck era and at such quantum domain Euclidean time plays the role of time. Thus it appears that the transition of the universe through the throat at $\tau = 0$ gives rise wormhole configuration (21), which naturally evolves (i.e. transits) to an inflationary era of power law expansion with the Lorentzian time. This appears as a transition from wormhole configuration to inflationary era. This is true in the action only with a term $\mathcal{R}^{\frac{3}{2}}$ in the modified theory of gravity.

7 Discussion

A study of Noether symmetry in the modified theory of gravity with $F(\mathcal{R})$ term in the action in 5-dimensional Kaluza–Klein cosmology yields $F(\mathcal{R}) \sim \mathcal{R}^n$ for arbitrary n (except $n = 1$), which is new in the literature. It is observed that the field equations are same both in the Lorentzian and Euclidean signature. Hence we consider solution of the Euclidean field equations for $n = \frac{3}{2}$, $n = 2$ and also consider solution of the Wheeler–DeWitt equation to study wormhole. Some of the important features are the followings:

1. We have an analytic solution of wormhole (Eq. 21) for $n = 3/2$ in a static internal space. Further wormhole configuration follows from a numerical solution in $n = 3/2$ with a dynamical internal space without dimensional reduction.
2. The field equations admits solution with $n = 2$ in static internal space, but it does not allow wormhole configuration. Wormhole solution is not attainable both in analytic and numerical method due to complexity of the field equations.
3. The wave function from the solution of the Wheeler–DeWitt equation in terms of old configuration space variables $\{a, b, \mathcal{R}\}$ allows wormhole for $n = 3/2$ and 2 satisfying Hawking–Page wormhole boundary conditions. It is important to note that $n = 2$ does not allow wormhole from the Euclidean field equations even for static internal space, however wormhole is allowed both in the Euclidean equation and Wheeler–DeWitt equation for $n = 3/2$.
4. The wave function [solution (40)] from the solution of the WD-equation in terms of variables $\{Q_1, Q_2, Q_3\}$ allows wormhole configuration for arbitrary n with $n > \frac{1}{2}$, except $n = 1$ and $\frac{13}{10}$. Further solution (40) agrees with (34) and (37) respectively for $n = 3/2$ and $n = 2$.
5. We recover the idea of time (instead of considering semi-classical approximation) by identifying the cyclic variable Q_3 as a time variable in canonical quantization with the notion of Noether constant of motion. As a consequence we have an idea of probability density form the continuity equation, which is not available from the WD-equation in general.
6. Finally we present an idea of transition from the wormhole configuration of the universe to the power law inflation asymptotically in $n = 3/2$.

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